Intérrogation du 08/02

Groupe 1

1/ (*) $y' + xy = e^{-x^2/2}$ → éq. diff. linéaire non-homogène, d'ordre 1

Eq. homogène associée:

$y' + xy = 0 \Rightarrow \frac{y'}{y} = -x \Rightarrow \ln y = -\frac{x^2}{2} + \text{const}$

Donc $y_{hom.} = C e^{-x^2/2}$ et on cherche la solution de (*) comme

$$y(x) = C(x) e^{-x^2/2}$$

$$\Downarrow$$

$$C' e^{-x^2/2} - x \cancel{C e^{-x^2/2}} + x \cancel{C e^{-x^2/2}} = e^{-x^2/2}$$

$$\Downarrow$$

$$C' = 1 \Rightarrow C(x) = x + D$$

Réponse générale: $y(x) = (x+D)e^{-x^2/2}$.

2/ $A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$

$\det(A - \lambda \mathbb{1}) = \det \begin{pmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 - 9$

(i) Les valeurs propres sont solutions de

$(2-\lambda)^2 - 9 = 0 \Rightarrow \lambda - 2 = \pm 3 \Rightarrow \lambda_1 = 5,\ \lambda_2 = -1.$

(ii) 1). $\lambda_1 = +5 \Rightarrow$ le vecteur propre associé vérifie $\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 5\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

$$\Downarrow$$

$$2\alpha + 3\beta = 5\alpha$$

$$\Downarrow$$

$$\alpha = \beta$$

et donc on peut poser $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

2. $\lambda_2 = -1 \Longrightarrow \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = -1 \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Rightarrow 2\alpha + 3\beta = -\alpha$

$$\Downarrow$$

$$\alpha = -\beta$$

et on peut choisir $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

iii). $e^{tA} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} =$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{5t} & 0 \\ 0 & e^{-t} \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} =$$

$$= \frac{1}{2} \begin{pmatrix} e^{5t} & e^{-t} \\ e^{5t} & -e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{5t}+e^{-t} & e^{5t}-e^{-t} \\ e^{5t}-e^{-t} & e^{5t}+e^{-t} \end{pmatrix}$$

Groupe 2

1] $y' + y \ln x = \sin x \, e^{x - x\ln x}$

L'équation homogène:

$y_0' + y_0 \ln x = 0 \Rightarrow \frac{y_0'}{y_0} = -\ln x \Rightarrow \ln y_0 = -\int \ln x \, dx =$
$= -(x\ln x - x) + \text{cnst}$

$y_0 = C e^{x - x\ln x}$

On cherche la solution comme

$y(x) = C(x) e^{x - x\ln x}$

$\Downarrow$

$C' e^{x-x\ln x} + \cancel{C e^{x-\ln x \cdot x}(-\ln x)} + \cancel{C e^{x-\ln x \cdot x}\ln x} =$
$= \sin x \, e^{x-x\ln x}$

$\Downarrow$

$C' = \sin x \Rightarrow C(x) = -\cos x + D$

$\Downarrow$

$y(x) = (D - \cos x) e^{x - x\ln x}$.

2] i). $A = \begin{pmatrix} 3 & 2+i \\ 2-i & 3 \end{pmatrix} \Rightarrow \det(A - \lambda \mathbb{1}) = \det \begin{pmatrix} 3-\lambda & 2+i \\ 2-i & 3-\lambda \end{pmatrix} =$
$= (3-\lambda)^2 - 5 \Longrightarrow \lambda_{1,2} = 3 \pm \sqrt{5}$

ii) $\begin{pmatrix} 3 & 2+i \\ 2-i & 3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \underset{\lambda_1}{\underbrace{(3+\sqrt{5})}} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Rightarrow \cancel{3\alpha} + (2+i)\beta = (\cancel{3} + \sqrt{5})\alpha$

$\Downarrow$

$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 2+i \\ \sqrt{5} \end{pmatrix}$

De façon analogue,

$$\begin{pmatrix} 3 & 2+i \\ 2-i & 3 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \underbrace{(3-\sqrt{5})}_{\lambda_2}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Longrightarrow \cancel{3}\alpha + (2+i)\beta = (\cancel{3}-\sqrt{5})\,\alpha$$

et donc on peut prendre le vecteur propre associé comme $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 2+i \\ -\sqrt{5} \end{pmatrix}$

iii).

$$e^{tA} = \begin{pmatrix} 2+i & 2+i \\ \sqrt{5} & -\sqrt{5} \end{pmatrix}\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}\begin{pmatrix} 2+i & 2+i \\ \sqrt{5} & -\sqrt{5} \end{pmatrix}^{-1} =$$

$$= -\frac{1}{2(2+i)\sqrt{5}} \begin{pmatrix} 2+i & 2+i \\ \sqrt{5} & -\sqrt{5} \end{pmatrix}\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}\begin{pmatrix} -\sqrt{5} & -(2+i) \\ -\sqrt{5} & 2+i \end{pmatrix} =$$

$$= -\frac{1}{2(2+i)\sqrt{5}} \begin{pmatrix} (2+i)e^{\lambda_1 t} & (2+i)e^{\lambda_2 t} \\ \sqrt{5}\, e^{\lambda_1 t} & -\sqrt{5}\, e^{\lambda_2 t} \end{pmatrix}\begin{pmatrix} -\sqrt{5} & -(2+i) \\ -\sqrt{5} & 2+i \end{pmatrix} =$$

$$= -\frac{1}{2(2+i)\sqrt{5}} \begin{pmatrix} -\sqrt{5}(2+i)(e^{\lambda_1 t}+e^{\lambda_2 t}) & -(2+i)^2(e^{\lambda_1 t}-e^{\lambda_2 t}) \\ -5(e^{\lambda_1 t}-e^{\lambda_2 t}) & -(2+i)\sqrt{5}(e^{\lambda_1 t}+e^{\lambda_2 t}) \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{e^{\lambda_1 t}+e^{\lambda_2 t}}{2} & \frac{2+i}{2\sqrt{5}}(e^{\lambda_1 t}-e^{\lambda_2 t}) \\ \frac{2-i}{2\sqrt{5}}(e^{\lambda_1 t}-e^{\lambda_2 t}) & \frac{e^{\lambda_1 t}+e^{\lambda_2 t}}{2} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{e^{(3+\sqrt{5})t}+e^{(3-\sqrt{5})t}}{2} & \frac{2+i}{2\sqrt{5}}\left(e^{(3+\sqrt{5})t}-e^{(3-\sqrt{5})t}\right) \\ \frac{2-i}{2\sqrt{5}}\left(e^{(3+\sqrt{5})t}-e^{(3-\sqrt{5})t}\right) & \frac{e^{(3+\sqrt{5})t}+e^{(3-\sqrt{5})t}}{2} \end{pmatrix}$$